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| Name of Course | : **CBCS B.Sc. (H) Mathematics** |
| Unique Paper Code | : **32351302_OC** |
| Name of Paper | : **C6-Group Theory-I** |
| Semester | : **III** |
| Duration | : **3 hours** |
| Maximum Marks | : **75 Marks** |

*Attempt any four questions. All questions carry equal marks.*

**1.** Let $G = \mathrm{GL}(n, \mathbb{R})$. Let $H = \{A \in G \mid \det A \text{ is a power of } 5\}$. Then prove or disprove that $H$ is a subgroup of $G$. Find the elements in $U(10)$ and $U(12)$ that satisfy the equation $x^2 = 1$.

**2.** List all the elements of order 3 in $\mathbb{Z}_{24}$. Find the smallest subgroup of $\mathbb{Z}$ containing 12 and 18. Determine the subgroup lattice for $\mathbb{Z}_{24}$.

**3.** Let $S_n$ be the symmetric group of degree $n$. Suppose that $\alpha \in S_n$ can be written as a product of disjoint cyclic permutations of lengths $m_1, m_2, \dots, m_r$, $(r \in \mathbb{N})$, respectively. Then prove that the order of $\alpha$ is $\mathrm{lcm}(m_1, m_2, \dots, m_r)$. Find the orders of (13)(27)(456)(8)(1237)(648)(5) and (124) (345). Furthermore, show that if $H$ is a subgroup of $S_n$ then either every member of $H$ is an even permutation or exactly half of them are even. Also, find $Z(S_n)$ for $n \geq 3$.

**4.** Show that for a finite group $G$, the index of a subgroup $H$ in $G$ is $|G|/|H|$. Prove that every subgroup of index 2 of a group $G$ is normal. Give an example of a subgroup $H$ of index 3 in a group $G$ which is not normal in $G$. Also, determine the index of $3\mathbb{Z}$ in $\mathbb{Z}$.

**5.** Let $H = \{\beta \in S_5 : \beta(1) = 1\}$ and $K = \{\beta \in S_5 : \beta(2) = 2\}$. Prove that $H$ is isomorphic to $K$. Is the same true if $S_5$ is replaced by $S_n$, where $n \geq 3$? Further prove or disprove that $S_4$ is isomorphic to $D_{12}$.

**6.** If $H$ is a subgroup of $G$ and $K$ is a normal subgroup of $G$, then prove that $H/(H \cap K)$ is isomorphic to $HK/K$. Also determine all homomorphisms from $\mathbb{Z}_n$ to itself.